



# Casimir energy in the MIT bag model

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## Abstract

The vacuum energies corresponding to massive Dirac fields with the boundary conditions of the MIT bag model are obtained. The calculations are done with the fields occupying the regions inside and outside the bag, separately. The renormalization procedure for each of the situations is studied in detail, in particular the differences occurring with respect to the case when the field extends over the whole space. The final result contains several constants undergoing renormalization, which can be determined only experimentally. The non-trivial finite parts which appear in the massive case are found exactly, providing a precise determination of the complete, renormalized zero-point energy for the first time, in the fermionic case. The vacuum energy behaves like inverse powers of the mass for large masses.

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# 1 I. Introduction

The first modern calculation of the vacuum energy density of a quantum field in the presence of boundaries is almost fifty years old. As is well known, it is due to H. Casimir [1]. Its first measurable consequence was the attraction in the electromagnetic vacuum of two neutral, infinitely conducting plates (thereafter called Casimir effect, see for instance Ref. [2]). There had been, before that time, other calculations and explanations of the attraction of two neutral bodies, understanding them as mere van der Waals effects, [3], and also a paper by Casimir and Polder [4] where the finiteness of the velocity of light had been taken into account. Casimir's paper [1] was the first one where an absolutely modern quantum field theoretical calculation was performed, using the concept of zero-point energy (whose physical relevance was somehow unclear at that time). The treatment of the divergences resulting from the infinitely many degrees of freedom has been the most difficult point. Since then, calculations of the vacuum energy have attracted the interest of many scientists because it turns out that, in different contexts, the inclusion of quantum fluctuations about semi-classical configurations is essential. On the other hand, spherically symmetrical situations are very important for practical applications. The calculations involved are certainly much more complicated than in the case of systems with plane boundaries.

Historically the first far reaching ideas involving vacuum energies in the case of spherical configurations also originated with Casimir. He proposed that the force stabilizing a classical electron model arises from the zero-point energy of the electromagnetic field within and without a perfectly conducting spherical shell [5]. Having found an attractive force between parallel plates due to the vacuum energy [1], the hope was that the same would occur for the spherically symmetric situation. Unfortunately, as Boyer [6] first showed, for this geometry the stress is repulsive [7, 8]. Nowadays it is known that the Casimir energy depends strongly on the geometry of the space-time and on the boundary conditions imposed. This is a very active field of research (see, for instance, [9],[10]).

More recently the zero-point energy has received considerable attention in the context of the bag model [11]-[15] and chiral bag model [16]-[22]. In these systems, quarks and gluons are free inside the bag, but are absolutely confined to it, being unable to cross the boundary surface. This is imposed, mathematically, by appropriate boundary conditions. The sum of the mesonic, valence quark and vacuum quark contributions to the baryonic number have been found to be independent of the bag radius and pion field strength, being the vacuum quark contributions—which are analogues of the Casimir effect in QED—essential in the calculation of baryonic observables. The issue of regularization in this model is certainly non-trivial. Under specific circumstances, different regularization procedures can yield different results and real physical problems arise in the calculation of quark vacuum contributions to some barionic observables, as the energy itself.

The Casimir energy for a spherical capacitor is usually given by the sum of two terms: an internal and an external one. Volume divergences cancel in each of the two regions separately, but surface and curvature divergences survive in each part and only cancel when interior and exterior contributions are added up, leaving then constant terms. In chiral bag models an additional divergence—logarithmic in the plasma frequency—appears. Having done these considerations, one should observe, however, that in the bag model quarks are supposed to exist in the *interior* of the bag only and, therefore, there is no clear way of eliminating these surface, curvature and logarithmic divergences. Some authors (see [16]) have suggested a revision of the Cheshire cat principles in the sense of including free quarks as correct high-energy degrees of freedom for the bag exterior, which would appear with a smooth transition at energies of the order of some GeV. In this way divergences would cancel quite naturally. However, in our opinion this deconfining transition is even a harder issue than those of regularization and renormalization. It will not be dealt with here.

For massless fermions the zero-point energy was considered some time ago in Refs. [23, 24], finite temperature effects were taken into account in [15, 21]. The massless fermionic field inside and outside the spherical bag was analysed in [24]. In the last case, a mutual cancellation of the divergences of the inner and outer spaces occurs. As a result, finite zero-point energies are found. However, when considering only the inner space, divergences arise and it is necessary to introduce contact terms and perform a renormalization of their coupling. Results for the massive fermionic fields contain new ultraviolet-divergent terms in addition to those occurring in the massless case, as has been discussed in [25]. Further considerations, especially on the renormalization procedure—necessary in order to carry out these calculations—and also on its precise physical interpretation, can be found in [26].

In most of the papers mentioned above a Green’s function approach has been used in order to calculate the zero-point energy. An exception is Ref. [26], where, in the general setting of an ultrastatic spacetime with or without boundaries, a systematic procedure which makes use of zeta function regularization was developed. In this approach, a knowledge of the zeta function of the operator associated with the field equation together with (eventually) some appropriate boundary conditions is needed. Recently, a detailed description of how to obtain the zeta function for a massive scalar field inside a ball satisfying Dirichlet or Robin boundary conditions has been given elsewhere by the authors of the present work [27, 28]. An analytical continuation to the whole complex plane was obtained there and subsequently applied to the computation of an arbitrary number of heat-kernel coefficients. In ensuing papers [29, 30] the functional determinant was considered too and, furthermore, the method has been also applied to spinors [31] and p-forms [33, 34, 35]. All the above considerations are purely analytical and quite precise. In order to obtain explicit values for the Casimir energy, however, a numerical evaluation of an integral was necessary. This has

been done in different cases, in particular for the massless scalar field and the electromagnetic field [37], partly reobtaining previous results.

To finish this description of recent previous work, let us mention that in Ref. [38] we have investigated the case of a massive scalar field in the bag. We have discussed there how, for the case of a massive field —already for a *scalar* one— non-trivial finite parts which depend on an adimensional variable involving the mass are present, that need to be properly renormalized, in order to get the corresponding zero-point energy. In the present paper we shall extend our analysis to the case of Dirac fields, generalizing in this way our considerations to a situation that approaches very much the conditions of a realistic MIT bag model.

The organization of the paper is as follows. We shall rely on our previous work (dealing with the bosonic case) for a precise description of our method —which was given there in full detail [38]— as well as for the particular formulas that are needed in the subsequent study of the zeta function of the problem we consider here. We feel that to repeat all this here would not be justified. Consequently, in Sect. 2 we will proceed already with the specific description of the model for the case of Dirac fields inside the bag with boundary conditions corresponding to the MIT bag. Starting from the Dirac equation and imposing the boundary conditions we will derive an eigenvalue equation in terms of Bessel functions. This will be the basic equation to solve, what we shall do in the same section for the interior of the bag. In section 3 we will explain the renormalization scheme used for the model. Sect. 4 contains the analogous treatment for the region exterior to the bag and for the whole space. Adding up the interior and the exterior contributions, we will see how the divergences cancel among themselves, as well as the influence of this cancellation on the compulsory renormalization process. It turns out that important differences with respect to the non-fermionic case appear concerning this issue, although we shall argue that, in the end, they will not affect substantially the interpretation of the physical results. Sect. 5 is devoted to conclusions. The appendixs contain some hints and technical details that have been used in the derivation of the zeta function (App. A) and a full list of the constituents that build up the subtraction terms in the decomposition of the zeta function, an essential (although rather technical) step in our method (App. B).

## 2 Fermions inside the bag

The first task is to derive the energy eigenvalue equations for a Dirac spinor subject to the MIT bag boundary conditions. The setting we consider first is the Dirac spinor inside a spherically symmetric bag confined to it by the appropriate boundary conditions. The coordinates we use are just the spherical ones,  $r, \theta, \varphi$ ,

which best adapt to the form of the bag. Thus, we must solve the equation:

$$H\phi_n(\vec{r}) = E_n\phi_n(\vec{r}), \quad (2.1)$$

$H$  being the Hamiltonian,

$$H = -i\gamma^0\gamma^\alpha\partial_\alpha + \gamma^0m, \quad (2.2)$$

with the boundary conditions

$$\left[1 + i\left(\frac{\vec{r}}{r}\vec{\gamma}\right)\right]\phi_n|_{r=R} = 0. \quad (2.3)$$

These boundary conditions guarantee that no quark current is lost through the boundary.

The separation to be carried out in the eigenvalue equation (2.1) is rather standard and will not be given here in detail. Let  $\vec{J}$  be the total angular momentum operator and  $K$  the spin projection operator. Then there exists a simultaneous set of eigenvectors of  $H, \vec{J}^2, J_3, K$  and the parity  $P$ . The eigenfunctions for positive eigenvalues  $\kappa = j + 1/2$  of  $K$  read

$$\phi_{jm} = \frac{A}{\sqrt{r}} \begin{pmatrix} iJ_{j+1}(\omega r)\Omega_{jlm}\left(\frac{\vec{r}}{r}\right) \\ -\sqrt{\frac{E-m}{E+m}}J_j(\omega r)\Omega_{jl'm}\left(\frac{\vec{r}}{r}\right) \end{pmatrix}, \quad (2.4)$$

whereas, for  $\kappa = -(j + 1/2)$ , one finds

$$\phi_{jm} = \frac{A}{\sqrt{r}} \begin{pmatrix} iJ_j(\omega r)\Omega_{jlm}\left(\frac{\vec{r}}{r}\right) \\ \sqrt{\frac{E-m}{E+m}}J_{j+1}(\omega r)\Omega_{jl'm}\left(\frac{\vec{r}}{r}\right) \end{pmatrix}. \quad (2.5)$$

Here  $\omega = \sqrt{E^2 - m^2}$ ,  $A$  is a normalization constant and  $\Omega_{jlm}(\vec{r}/r)$  are the well known spinor harmonics. In order to obtain eigenfunctions of the parity operator we must set  $l' = l - 1$  in (2.4) and  $l' = l + 1$  in (2.5). In both cases,  $j = 1/2, 3/2, \dots, \infty$ , and the eigenvalues are degenerate in  $m = -j, \dots, +j$ .

Imposing the boundary conditions (2.3) on the solutions (2.4) and (2.5), respectively, one easily finds the corresponding implicit eigenvalue equation. For  $\kappa > 0$ , it reads

$$\sqrt{\frac{E+m}{E-m}}J_{j+1}(\omega R) + J_j(\omega R) = 0, \quad (2.6)$$

and for  $\kappa < 0$ , on its turn,

$$J_j(\omega R) - \sqrt{\frac{E-m}{E+m}}J_{j+1}(\omega R) = 0. \quad (2.7)$$

Regretfully, it is not possible to find an explicit solution of equations (2.6) and (2.7). But as we have shown in our previous paper for the case of the scalar field—and will describe below for the spinor field—the information displayed in (2.6) and (2.7) is already enough for the calculation of the ground state energy for massive spinors in the bag.

The regularization of this ground state energy will be performed by using the zeta function method. In short, we consider

$$\begin{aligned} E_0(s) &= -\frac{1}{2} \sum_k \left(E_k^2\right)^{1/2-s} \mu^{2s}, \quad \text{Re } s > s_0 = 2 \\ &= -\frac{1}{2} \zeta^{(int)}(s-1/2) \mu^{2s}, \end{aligned} \quad (2.8)$$

and later analytically continue to the value  $s = 0$  in the complex plane. Here  $s_0$  is the abscissa of convergence of the series,  $\mu$  the usual mass parameter and

$$\zeta^{(int)}(s) = \sum_k (E_k^2)^{-s}. \quad (2.9)$$

The power of the method lies in the well defined prescriptions and procedures that we have at our hand to analytically continue the series to the rest of the complex  $s$ -plane, even when the spectrum  $E_k$  is not known explicitly (as will in fact be the case). These procedures have been developed and described in great detail in [27, 28, 38], so that we can be brief here.

The zeta function in the interior space is given by

$$\begin{aligned} \zeta^{(int)}(s) &= 2 \sum_{j=1/2, 3/2, \dots}^{\infty} (2j+1) \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \\ &\quad \times \frac{\partial}{\partial k} \ln \left[ J_j^2(kR) - J_{j+1}^2(kR) + \frac{2m}{k} J_j(kR) J_{j+1}(kR) \right]. \end{aligned} \quad (2.10)$$

Here the factor of 2 results from taking into account particles and antiparticles. Using the method—ordinarily employed in this situation—of deforming the contour which originally encloses the singular points on the real axis, until it covers the imaginary axis, after simple manipulations we obtain the following equivalent expression for  $\zeta^{(int)}$ :

$$\begin{aligned} \zeta^{(int)}(s) &= \frac{2 \sin \pi s}{\pi} \sum_{j=1/2, 3/2, \dots}^{\infty} (2j+1) \int_{mR/j}^{\infty} dz \left[ \left( \frac{zj}{R} \right)^2 - m^2 \right]^{-s} \\ &\quad \times \frac{\partial}{\partial z} \ln \left\{ z^{-2j} \left[ I_j^2(zj) \left( 1 + \frac{1}{z^2} - \frac{2mR}{z^2 j} \right) + I_j^2(zj) \right. \right. \\ &\quad \left. \left. + \frac{2R}{zj} \left( m - \frac{j}{R} \right) I_j(zj) I_j'(zj) \right] \right\}. \end{aligned} \quad (2.11)$$

As is usual, we will now split the zeta function into two parts:

$$\zeta^{(int)}(s) = Z_N(s) + \sum_{i=-1}^N A_i(s), \quad (2.12)$$

namely a regular one,  $Z_N$ , and a remainder that contains the contributions of the  $N$  first terms of the Bessel functions  $I_\nu(k)$  as  $\nu, k \rightarrow \infty$  with  $\nu/k$  fixed [39]. The number  $N$  of terms that have to be subtracted is in general the minimal one necessary in order to absorb all possible divergent contributions into the groundstate energy, Eq. (2.8). In our case,  $N = 3$ . This is a general procedure, commonly applied in order to deal with such kind of divergences. We get

$$\begin{aligned} Z_3(s) &= 2 \frac{\sin \pi s}{\pi} \sum_{j=\frac{1}{2}}^{\infty} (2j+1) \int_{\frac{mR}{j}}^{\infty} dz \left[ \left( \frac{zj}{R} \right)^2 - m^2 \right]^{-s} \\ &\times \frac{\partial}{\partial z} \left\{ \ln \left[ I_j^2(zj) \left( 1 + \frac{1}{z^2} - \frac{2mR}{z^2 j} \right) + I_j^2(zj) + \frac{2R}{zj} \left( m - \frac{j}{R} \right) I_j(zj) I_j'(zj) \right] \right. \\ &\quad \left. - \ln \left[ \frac{e^{2j\eta} (1+z^2)^{\frac{1}{2}} (1-t)}{\pi j z^2} \right] - \sum_{k=1}^3 \frac{D_k(mR, t)}{j^k} \right\}, \end{aligned} \quad (2.13)$$

where  $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$  and  $t = 1/\sqrt{1+z^2}$ . After renaming  $mR = x$ , the relevant polynomials are given by

$$\begin{aligned} D_1(t) &= \frac{t^3}{12} + (x - 1/4)t \\ D_2(t) &= -\frac{t^6}{8} - \frac{t^5}{8} + \left( -\frac{x}{2} + 1/8 \right) t^4 + \left( -\frac{x}{2} + 1/8 \right) t^3 - \frac{t^2 x^2}{2} \\ D_3(t) &= \frac{179 t^9}{576} + \frac{3 t^8}{8} + \left( -\frac{23}{64} + \frac{7x}{8} \right) t^7 + (x - 1/2) t^6 + \left( \frac{9}{320} - \frac{x}{4} + \frac{x^2}{2} \right) t^5 \\ &\quad + \left( \frac{x^2}{2} + 1/8 - \frac{x}{2} \right) t^4 + \left( -\frac{x}{8} + \frac{5}{192} + \frac{x^3}{3} \right) t^3. \end{aligned} \quad (2.14)$$

The asymptotic contributions  $A_i(s)$ ,  $i = -1, \dots, 3$ , are defined as

$$\begin{aligned} A_{-1}(s) &= \frac{8 \sin(\pi s)}{\pi} \sum_{j=1/2}^{\infty} j(j+1/2) \int_{mR/j}^{\infty} \left( \left( \frac{xj}{R} \right)^2 - m^2 \right)^{-s} \frac{\sqrt{1+x^2} - 1}{x} \\ A_0(s) &= \frac{4 \sin(\pi s)}{\pi} \sum_{j=1/2}^{\infty} (j+1/2) \int_{mR/j}^{\infty} \left( \left( \frac{xj}{R} \right)^2 - m^2 \right)^{-s} \frac{\partial}{\partial x} \ln \frac{\sqrt{1+x^2} (1-t)}{x^2} \\ A_i(s) &= \frac{4 \sin(\pi s)}{\pi} \sum_{j=1/2}^{\infty} (j+1/2) \int_{mR/j}^{\infty} \left( \left( \frac{xj}{R} \right)^2 - m^2 \right)^{-s} \frac{\partial}{\partial x} \frac{D_i(t)}{j^i}. \end{aligned} \quad (2.15)$$



The small mass expansion can be conveniently represented as

$$\begin{aligned}
A_{-1}(s) &= \frac{R^{2s}}{\sqrt{\pi} \Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (mR)^{2k} \frac{\Gamma(k+s-\frac{1}{2})}{k+s} \\
&\quad \times \left[ 2\zeta(2k+2s-2, \frac{1}{2}) + \zeta(2k+2s-1, \frac{1}{2}) \right] \\
A_0(s) &= -\frac{R^{2s}}{2\sqrt{\pi} \Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (mR)^{2k} \frac{\Gamma(k+s+\frac{1}{2})}{k+s} \\
&\quad \times \left( 2\zeta(2k+2s-1, \frac{1}{2}) + \zeta(2k+2s, \frac{1}{2}) \right) \\
A_i(s) &= -\frac{2R^{2s}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (mR)^{2k} \left[ 2\zeta(2k+2s+i-1, \frac{1}{2}) + \zeta(2k+2s+i, \frac{1}{2}) \right] \\
&\quad \times \sum_{a=0}^{2i} x_{i,a} \frac{\Gamma(k+s+\frac{a+i}{2})}{\Gamma(\frac{a+i}{2})}. \tag{2.16}
\end{aligned}$$

In this expression, the  $x_{i,a}$  are the coefficients of the expansion of the functions  $D_i(t)$ , i.e.,

$$D_i(t) = \sum_{a=0}^{2i} x_{i,a} t^{a+i}. \tag{2.17}$$

Note that here we encounter the same problem that occurred already in the scalar case. One needs a representation that is useful and valid for an (in principle) arbitrary value of  $m$ . To this end one can actually proceed in different ways, casting the final result in terms of convergent series or integrals. Our *leitmotiv* will be the following: we will always try to express the final result in terms of the formula which is more appropriate for practical evaluation (e.g., numerical, in general). This means that, sometimes, instead of having the closed convergent sums that could be universally used in the scalar case, rapidly converging integrals better suited for numerical analysis will be here preferred.

With this aim, we note that after performing the  $z$ -integration the  $A_i(s)$ , for  $i \geq 1$ , can be written in the following form,

$$\begin{aligned}
A_i(s) &= -\frac{4m^{-2s}}{\Gamma(s)} \sum_{a=0}^{2i} \frac{x_{i,a}}{(mR)^{i+a}} \frac{\Gamma(s+(i+a)/2)}{\Gamma((i+a)/2)} \\
&\quad \times \left[ f(s; 1+a; (i+a)/2) + \frac{1}{2} f(s; a; (i+a)/2) \right], \tag{2.18}
\end{aligned}$$

with the definition

$$f(s; a; b) = \sum_{\nu=1/2, 3/2, \dots}^{\infty} \nu^a \left( 1 + \left( \frac{\nu}{mR} \right)^2 \right)^{-s-b}. \tag{2.19}$$

The remaining thing to do in the present case is to calculate the  $f(s; a; b)$  for the relevant values at  $s = -1/2$ . This is a systematic calculation that will be sketched in App. A. Let us here mention only that an essential step is to use the simple recurrence:

$$f(s; a; b) = (mR)^2 [f(s; a - 2; b - 1) - f(s; a - 2; b)]. \quad (2.20)$$

In App. B we give the whole list of starting terms that, in addition to the recurrence formula (2.20), are strictly necessary for obtaining explicitly all the  $A_i(s)$  needed in our calculation.

### 3 Discussion of the renormalization

For the discussion of the renormalization let us look for the divergent terms in the groundstate energy. By construction they are all contained in the contributions  $A_i(s)$ . Having their explicit form at hand (see Apps. A and B) they can be given quickly. In particular, we have, for the interior part:

$$\begin{aligned} \text{res } A_{-1}^{(int)}(-1/2) &= -\frac{m^4 R^3}{6\pi} + \frac{m^2 R}{12\pi} + \frac{7}{480\pi R} \\ \text{res } A_0^{(int)}(-1/2) &= -\frac{m^2 R}{2\pi} - \frac{1}{24\pi R} \\ \text{res } A_1^{(int)}(-1/2) &= -\frac{m^3 R^2}{\pi} + \frac{m^2 R}{12\pi} + \frac{m}{12\pi} - \frac{1}{48\pi R} \\ \text{res } A_2^{(int)}(-1/2) &= -\frac{m^2 R}{4} - m \left( \frac{1}{8} + \frac{1}{2\pi} \right) + \frac{1}{128R} + \frac{1}{24\pi R} \\ \text{res } A_3^{(int)}(-1/2) &= \frac{2m^3 R^2}{3\pi} + m^2 R \left( \frac{1}{4} + \frac{2}{3\pi} \right) \\ &\quad + m \left( \frac{1}{8} + \frac{7}{20\pi} \right) - \frac{1}{128R} - \frac{97}{10080\pi R} \end{aligned}$$

and, as a result,

$$\text{Res } \zeta^{(int)}(-1/2) = -\frac{1}{63\pi R} - \frac{m}{15\pi} + \frac{m^2 R}{3\pi} - \frac{m^3 R^2}{3\pi} - \frac{m^4 R^3}{6\pi}. \quad (3.1)$$

These terms form the minimal set of counterterms necessary in order to renormalize our theory.

In the scalar case one had the peculiar situation that there were no divergent contributions of the form  $\sim m^3$ ,  $m$  in the zeta function description [38] —although in other regularizations they indeed appear [26]. So in principle one had the choice of renormalizing the associated couplings. Contrarily, for spinors, as seen in (3.1), the coupling constants of *all* terms appearing have to be renormalized. We are led into a physical system consisting of two parts:

1. A classical system consisting of a spherical surface ('bag') with radius  $R$ . Its energy reads:

$$E_{class} = pV + \sigma S + FR + k + \frac{h}{R}, \quad (3.2)$$

where  $V = \frac{4}{3}\pi R^3$  and  $S = 4\pi R^2$  are the volume and the surface of the bag, respectively. This energy is determined by the parameters:  $p$  pressure,  $\sigma$  surface tension, and  $F$ ,  $k$ , and  $h$  which do not have special names.

2. A spinor quantum field  $\varphi(x)$  obeying the Dirac equation and the MIT boundary conditions (2.3) on the surface. The quantum field has a ground state energy given by  $E_0$ , Eq. (2.8).

Thus, the complete energy of the physical system is

$$E = E_{class} + E_0 \quad (3.3)$$

and in this context the renormalization can be achieved by shifting the parameters in  $E_{class}$  by an amount which cancels the divergent contributions.

First we perform a kind of minimal subtraction, where only the divergent contribution is eliminated,

$$\begin{aligned} p &\rightarrow p - \frac{m^4}{16\pi^2} \frac{1}{s} & \sigma &\rightarrow \sigma - \frac{m^3}{24\pi^2} \frac{1}{s} \\ F &\rightarrow F + \frac{m^2}{6\pi} \frac{1}{s} & k &\rightarrow k - \frac{m}{30\pi} \frac{1}{s} \\ h &\rightarrow h - \frac{1}{126\pi} \frac{1}{s}. \end{aligned} \quad (3.4)$$

The quantities  $\alpha = \{p, \sigma, F, k, h\}$  are a set of free parameters of the theory to be determined experimentally. In principle we are free to perform finite renormalizations at our choice of all the parameters.

We find natural to perform two further renormalizations. First it is possible to determine the asymptotic behavior of the  $A_i$  for  $m \rightarrow \infty$  using the results of Apps. A and B. The finite pieces not vanishing in the limit  $m \rightarrow \infty$  are all of the same type appearing in the classical energy. Our first finite renormalization is such that those pieces are cancelled. As a result, only the "quantum contributions" are finally included, because, physically, a quantum field of infinite mass is not expected to fluctuate. The resulting  $A_i$  will be called  $A_i^{(ren)}$ .

Concerning  $Z_3$  we have not been able to determine analytically its complete non-vanishing behavior for  $m \rightarrow \infty$ . Instead, for the numerical analysis, as shown in Fig. 1, we have constructed a numerical fit of  $Z_3$  by a polynomial of the form

$$P(m) = \sum_{i=0}^4 c_i m^i,$$

and then subtracted this polynomial from  $Z_3$ . As explained above, this is nothing else than an ulterior finite renormalization. The result will be denoted by  $Z_3^{(ren)}$ .

Summing up, we can write the complete energy as

$$E = E_{class} + E_0^{(ren)}, \quad (3.5)$$

where  $E_{class}$  is defined as in (3.2) with the renormalized parameters  $\alpha$  and  $E_0^{(ren)} = Z_3^{(ren)} + \sum_{i=-1}^3 A_i^{(ren)}$ .

Figure 1 shows the numerical analysis of the energy  $E_0^{(ren)}$  of the system for this specific choice of renormalization. The energy exhibits a clear minimum corresponding to a stability bag radius.

## 4 Exterior of the bag and a model for the whole space

The analysis of the region exterior to the bag is quite similar to the one carried out for the interior region. Only some specific differences appear both in the formulas and in the results. The expression for the zeta function in the exterior region is essentially the same as the one corresponding to the interior, but for the replacement of the Bessel  $I_j$  functions with Bessel  $K_j$  functions, namely

$$\begin{aligned} \zeta^{(ext)}(s) &= \frac{2 \sin \pi s}{\pi} \sum_{j=1/2, 3/2, \dots}^{\infty} (2j+1) \int_{mR/j}^{\infty} dz \left( (zj/R)^2 - m^2 \right)^{-s} \\ &\quad \times \frac{\partial}{\partial z} \ln \left[ z^{2j} \left( K_j^2(zj) - K_{j+1}^2(zj) + \frac{2mR}{zj} K_j(zj) K_{j+1}(zj) \right) \right]. \end{aligned}$$

The splitting of the zeta function has also the same aspect as for the interior region. We have, in particular

$$A_{-1}^{(ext)}(s) = A_{-1}^{(int)}(s),$$

$$\begin{aligned} A_0^{(ext)}(s) &= \frac{4 \sin(\pi s)}{\pi} \sum_{j=1/2}^{\infty} (j+1/2) \int_{mR/j}^{\infty} dz \left( (zj/R)^2 - m^2 \right)^{-s} \\ &\quad \times \frac{\partial}{\partial z} \ln \left[ \frac{1+t}{t} \right], \end{aligned}$$

and the polynomials that replace the  $D_i(t)$  above are here ( $x = mR$ )

$$\begin{aligned} \overline{D}_1(t) &= \frac{t}{4} + x t - \frac{t^3}{12}, \\ \overline{D}_2(t) &= \frac{-x^2 t^2}{2} - \frac{t^3}{8} - \frac{x t^3}{2} + \frac{t^4}{8} + \frac{x t^4}{2} + \frac{t^5}{8} - \frac{t^6}{8}, \end{aligned}$$

$$\begin{aligned}\overline{D}_3(t) = & \frac{-5t^3}{192} - \frac{xt^3}{8} + \frac{x^3t^3}{3} + \frac{t^4}{8} + \frac{xt^4}{2} + \frac{x^2t^4}{2} - \frac{9t^5}{320} \\ & - \frac{xt^5}{4} - \frac{x^2t^5}{2} - \frac{t^6}{2} - xt^6 + \frac{23t^7}{64} + \frac{7xt^7}{8} + \frac{3t^8}{8} - \frac{179t^9}{576}.\end{aligned}$$

As for the functions  $A_i^{(ext)}(s)$ , one obtains the same expressions as before, with the replacement of the polynomials  $D_i(t)$  with the corresponding polynomials  $\overline{D}_i(t)$ .

In principle, the same procedure as before can be applied now in order to get an analytical expression for the whole energy of the exterior space. Instead, we want to restrict ourselves here to the specific changes that show up when discussing the renormalization. For doing that, we only have to consider the pole of the different  $A_i^{(ext)}$ . In particular, we have for the residua

$$\begin{aligned}\text{res } A_{-1}^{(ext)}(-1/2) &= -\frac{m^4 R^3}{6\pi} + \frac{m^2 R}{12\pi} + \frac{7}{480\pi R} = \text{res } A_{-1}^{(int)}(-1/2) \\ \text{res } A_0^{(ext)}(-1/2) &= \frac{m^2 R}{2\pi} + \frac{1}{24\pi R} = -\text{res } A_0^{(int)}(-1/2) \\ \text{res } A_1^{(ext)}(-1/2) &= -\frac{m^3 R^2}{\pi} - \frac{m^2 R}{12\pi} + \frac{m}{12\pi} + \frac{1}{48\pi R} \\ \text{res } A_2^{(ext)}(-1/2) &= -\frac{m^2 R}{4} + m\left(\frac{1}{8} - \frac{1}{2\pi}\right) + \frac{1}{128R} - \frac{1}{24\pi R} \\ \text{res } A_3^{(ext)}(-1/2) &= \frac{2m^3 R^2}{3\pi} + m^2 R\left(\frac{1}{4} - \frac{2}{3\pi}\right) \\ &\quad - m\left(\frac{1}{8} - \frac{7}{20\pi}\right) - \frac{1}{128R} + \frac{97}{10080\pi R}.\end{aligned}$$

This yields for the residue of the whole zeta function at the exterior region:

$$\text{Res } \zeta^{(ext)}(-1/2) = \frac{1}{63\pi R} - \frac{m}{15\pi} - \frac{m^2 R}{3\pi} - \frac{m^3 R^2}{3\pi} + \frac{m^4 R^3}{6\pi}. \quad (4.1)$$

Thus the minimal set of counterterms necessary in order to renormalize the theory in the exterior of the bag is identical to the one in the interior of the bag. The classical system is again described by Eq. (3.2).

The opposite sign of the coefficients in the divergences (3.1) and (4.1) corresponding to the odd powers of  $R$  can be easily explained by means of differential geometrical arguments, just observing that the curvature of the surface of the bag has opposite sign when looked at from the exterior and from the interior of the bag.

Contrary to the scalar case, the divergences from the two sides do not annihilate when adding up the two contributions. In fact, for the zeta function corresponding to the whole space (internal and external to the bag) we obtain:

$$\text{Res } \zeta(-1/2) = \text{Res } \zeta^{(int)}(-1/2) + \text{Res } \zeta^{(ext)}(-1/2) = -\frac{2m}{15\pi} - \frac{2m^3 R^2}{3\pi}, \quad (4.2)$$

therefore, the two free parameters  $\sigma$  and  $k$  remain even if the whole space is considered. The only exception is the massless field where for this reason the issue of renormalization is much simpler than in our case.

## 5 Conclusions

In this paper we have studied in considerable detail a quantum field theoretical system of a Dirac field with boundary conditions corresponding to the MIT bag model. This is the most natural continuation—in the direction towards approaching truly realistic physical systems—of previous work where only scalar fields were treated [38]. We have seen that the consideration of a fermionic field carries along a number of additional difficulties, mainly in relation with the philosophy of the renormalization process. Up to this point, the application of our techniques can be carried out essentially in the same way as for the scalar case. Starting from the Dirac equation and imposing the boundary conditions we have derived an eigenvalue equation in terms of Bessel functions. This basic expression has then been solved, implicitly, in the regions interior and exterior to the bag surface, by using contour integration. This has yielded the corresponding zeta function in each of the two domains. Extraction of the singular part of the zeta function has been also done exactly. However, adding up the contributions of the two parts, not all divergences cancel among themselves (as was the case for a scalar field), what theoretically influences the playground of the ulterior renormalization process.

In the end, a detailed consideration shows that, after renormalization, two dimensionless parameters remain whose values cannot be fixed theoretically, but have to be numerically adjusted by direct comparison with the physical system described by the model. In this, we must confess, we are still a bit far from our final goal. In the sense that, as it stands, our model cannot be considered yet to describe a realistic physical situation. This must be left to future work, given the complexity of the proposal. In any case, we should like to point out the rigour and strict systematicity of the approach we have used here, and also its relative simplicity, if we compare it with other methods of similar strength and generality.

As noticeable results of our analysis we would like to mention, that the Casimir energy may have a clear minimum associated with a stable bag radius (see Fig. 1). Comparing this behaviour with the one corresponding to the scalar field, where a maximum occurred, this difference can clearly be traced back to the anticommuting nature of the spinor fields, which shows up as a sign in the definition of the groundstate energy.

Another interesting observation is that, contrary to the case of parallel plates, the behaviour of the Casimir energy for large values of  $mR$  is not exponentially damped. Instead, as is clearly observed from the representations of the  $A_i(s)$  given in the Apps. A and B, we find a behavior in inverse powers of the mass.

This is directly connected with the nonvanishing of the extrinsic curvature at the bag.

Possible continuation of our approach is in the direction of finite temperature and finite densities as considered already for massless fermions in [15, 21, 22]. A natural question to ask concerns the possible appearance of a first order phase transition from an hadronic bag to a deconfined quark-gluon plasma within our framework. This is left for future work.

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## A Appendix: Explicit representations for the asymptotic contributions inside the bag

The essential formulas for the basic series  $f(s; a; b)$ , eq. (2.19), in the calculation are the following:

$$\begin{aligned} \sum_{\nu=1/2, 3/2, \dots}^{\infty} \nu^{2n+1} \left(1 + \left(\frac{\nu}{x}\right)^2\right)^{-s} &= \frac{1}{2} \frac{n! \Gamma(s-n-1)}{\Gamma(s)} x^{2n+2} \quad (\text{A.1}) \\ &+ (-1)^n 2 \int_0^x d\nu \frac{\nu^{2n+1}}{1 + e^{2\pi\nu}} \left(1 - \left(\frac{\nu}{x}\right)^2\right)^{-s} \\ &+ (-1)^n 2 \cos(\pi s) \int_x^{\infty} d\nu \frac{\nu^{2n+1}}{1 + e^{2\pi\nu}} \left(\left(\frac{\nu}{x}\right)^2 - 1\right)^{-s}, \end{aligned}$$

$$\begin{aligned} \sum_{\nu=1/2, 3/2, \dots}^{\infty} \nu^{2n} \left(1 + \left(\frac{\nu}{x}\right)^2\right)^{-s} &= \frac{1}{2} \frac{\Gamma(n+1/2) \Gamma(s-n-1/2)}{\Gamma(s)} x^{2n+1} \quad (\text{A.2}) \\ &- (-1)^n 2 \sin(\pi s) \int_x^{\infty} d\nu \frac{\nu^{2n}}{1 + e^{2\pi\nu}} \left(\left(\frac{\nu}{x}\right)^2 - 1\right)^{-s}. \end{aligned}$$

Using partial integrations one can get representations valid for values of  $s$  needed for the  $A_i(s)$ . One gets, for example, the following expansions around  $s = -1/2$ :

$$\sum_{\nu=1/2, 3/2, \dots}^{\infty} \nu^3 \left(1 + \left(\frac{\nu}{x}\right)^2\right)^{-s-3/2} = \frac{1}{2} \frac{\Gamma(s-1/2)}{\Gamma(s+3/2)} x^4$$

$$-x^2 \int_0^\infty d\nu \frac{d}{d\nu} \left[ \frac{\nu^2}{1 + e^{2\pi\nu}} \right] \ln |\nu^2 - x^2| + \mathcal{O}(s + 1/2),$$

$$\sum_{\nu=1/2, 3/2, \dots}^\infty \nu^2 \left( 1 + \left( \frac{\nu}{x} \right)^2 \right)^{-s-3/2} = -\frac{\pi}{2} x^3 + \pi \frac{x^3}{1 + e^{2\pi x}} + \mathcal{O}(s + 1/2)$$

showing clearly that one can obtain quickly convergent integrals, respectively expressions for the effective numerical evaluation of the involved sums.

All the particular values that are necessary to give the  $A_i(s)$ ,  $i = 1, 2, 3$ , explicitly (in addition to the recurrence (2.20)) are listed in App. B.

The first two leading asymptotic contributions,  $A_{-1}$  and  $A_0$  have to be treated in a slightly different way, as has been explained in detail in [38]. For completeness we give the final results

$$\begin{aligned} A_{-1}(s) = & \left( \frac{1}{s + 1/2} - \ln m^2 \right) \left( -\frac{R^3 m^4}{12\pi} + \frac{m^2 R}{24\pi} + \frac{7}{960\pi R} \right) \\ & + \frac{R^3 m^4}{24\pi} (1 - 4 \ln 2) - \frac{m^3 R^2}{6} + \frac{m^2 R}{24\pi} [2 \ln(2mR) - 1] + \frac{7}{960\pi R} [1 + 2 \ln(2mR)] \\ & - \frac{2}{\pi R} \int_0^\infty \frac{d\nu \nu}{1 + e^{2\pi\nu}} (\nu^2 - m^2 R^2) \ln |\nu^2 - m^2 R^2| \\ & - \frac{4m^2 R}{\pi} \int_0^\infty \frac{d\nu \nu}{1 + e^{2\pi\nu}} \left( \ln |\nu^2 - m^2 R^2| + \frac{\nu}{mR} \ln \left| \frac{mR + \nu}{mR - \nu} \right| \right) \quad (\text{A.3}) \\ & + \frac{m^2 R}{2\pi} \ln (1 + e^{-2\pi mR}) - \frac{1}{R} \int_{mR}^\infty \frac{d\nu \nu^2}{1 + e^{2\pi\nu}} - \frac{m^2 R}{\pi} \int_0^1 dy \ln (1 + e^{-2\pi mRy}), \end{aligned}$$

and

$$\begin{aligned} A_0(s) = & - \left( \frac{1}{s + 1/2} - \ln m^2 \right) \left( \frac{1}{48\pi R} + \frac{m^2 R}{4\pi} \right) \\ & + \frac{m^3 R^2}{6} + \frac{m^2 R}{\pi} \left[ \frac{5}{4} - \frac{1}{2} \ln 2 - \ln(mR) \right] - \frac{\ln 2}{24\pi R} \\ & - \frac{2}{R} \int_{mR}^\infty \frac{d\nu \nu^2}{1 + e^{2\pi\nu}} - m^3 R^2 \int_0^1 \frac{dx}{1 + e^{2\pi mR\sqrt{x}}} \\ & + \frac{1}{\pi R} \int_0^\infty \frac{d\nu \nu}{1 + e^{2\pi\nu}} \ln \left| 1 - \left( \frac{\nu}{mR} \right)^2 \right| \\ & - \frac{m^2 R}{2\pi} \int_0^\infty d\nu \left( \frac{d}{d\nu} \frac{1}{1 + e^{2\pi\nu}} \right) \int_0^1 \frac{dx}{\sqrt{x}} \ln |m^2 R^2 x - \nu^2|. \quad (\text{A.4}) \end{aligned}$$

This completes the description of our procedures to obtain numerically evaluable representations for all the  $A_i$ 's, needed for the calculation of the Casimir energy of the spinor inside the bag.



## B Appendix: Full list of constituent terms $f(a; b)$ to be used in addition to the recurrence formula

To simplify the expressions, we shall here use  $x$  for  $mR$ . In addition to the above recurrence, in order to determine all the  $A_i(s)$  explicitly one needs the following  $f(a; b)$ 's [we shall use the notation  $f(a; b) = f(-1/2; a; b)$ ]:

$$\begin{aligned}
f(0; 1/2) &= 0, & f(1; 1/2) &= -\frac{1}{2}x^2 + \frac{1}{24} \\
\frac{d}{ds} \Big|_{s=-1/2} f(s; 0; 1/2) &= -\pi x - 2\pi \int_x^\infty d\nu \frac{1}{1 + e^{2\pi\nu}} \\
\frac{d}{ds} \Big|_{s=-1/2} f(s; 1; 1/2) &= -\frac{1}{2}x^2 - 2 \int_0^\infty d\nu \frac{\nu}{1 + e^{2\pi\nu}} \ln \left| 1 - \left(\frac{\nu}{x}\right)^2 \right| \\
f(0; 1) &= \frac{x}{2(s+1/2)} + x \ln 2 + 2x^2 \int_x^\infty d\nu \frac{d}{d\nu} \left( \frac{1}{\nu(1 + e^{2\pi\nu})} \right) \left[ \left(\frac{\nu}{x}\right)^2 - 1 \right]^{1/2}, \\
f(0; 3/2) &= \frac{\pi x}{2} - \frac{\pi x}{1 + e^{2\pi x}}, \\
f(1; 3/2) &= \frac{x^2}{2(s+1/2)} + x^2 \ln x + x^2 \int_0^\infty d\nu \left( \frac{d}{d\nu} \frac{1}{1 + e^{2\pi\nu}} \right) \ln |\nu^2 - x^2|, \\
f(1; 1) &= 2x^2 \int_0^x d\nu \left( \frac{d}{d\nu} \frac{1}{1 + e^{2\pi\nu}} \right) \left[ 1 - \left(\frac{\nu}{x}\right)^2 \right]^{1/2}, \\
f(1; 2) &= -2x^2 \int_0^x d\nu \left( \frac{d}{d\nu} \frac{1}{1 + e^{2\pi\nu}} \right) \left| 1 - \left(\frac{\nu}{x}\right)^2 \right|^{-1/2}, \\
f(2; 2) &= \frac{x^3}{2(s+1/2)} + (\ln 2 - 1)x^3 + 2x^4 \int_x^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu}{1 + e^{2\pi\nu}} \right) \right] \left[ \left(\frac{\nu}{x}\right)^2 - 1 \right]^{1/2}, \\
f(2; 5/2) &= \frac{\pi}{4}x^3 - \frac{\pi}{2}x^4 \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu}{1 + e^{2\pi\nu}} \right) \Big|_{\nu=x}, \\
f(3; 5/2) &= \frac{x^4}{2(s+1/2)} + (\ln x - 1/2)x^4 + \frac{x^4}{2} \int_0^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^2}{1 + e^{2\pi\nu}} \right) \right] \ln |\nu^2 - x^2|, \\
f(3; 3) &= -\frac{2}{3}x^4 \int_0^x d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^2}{1 + e^{2\pi\nu}} \right) \right] \left[ 1 - \left(\frac{\nu}{x}\right)^2 \right]^{-1/2}, \\
f(4; 3) &= \frac{x^5}{2(s+1/2)} + (3 \ln 2 - 4) \frac{x^5}{3} \\
&\quad + \frac{2x^6}{3} \int_x^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^3}{1 + e^{2\pi\nu}} \right) \right] \left[ \left(\frac{\nu}{x}\right)^2 - 1 \right]^{1/2}, \\
f(4; 7/2) &= \frac{3\pi}{16}x^5 - \frac{\pi}{8}x^6 \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^3}{1 + e^{2\pi\nu}} \right) \Big|_{\nu=x},
\end{aligned}$$

$$\begin{aligned}
f(5; 7/2) &= \frac{x^6}{2(s+1/2)} + (\ln x - 3/4)x^6 \\
&\quad + \frac{x^6}{8} \int_0^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^4}{1+e^{2\pi\nu}} \right) \right] \ln |\nu^2 - x^2|, \\
f(5; 4) &= -\frac{2x^6}{15} \int_0^x d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^4}{1+e^{2\pi\nu}} \right) \right] \left[ 1 - \left( \frac{\nu}{x} \right)^2 \right]^{-1/2}, \\
f(6; 4) &= \frac{x^7}{2(s+1/2)} + (\ln 2 - 23/15)x^7 \\
&\quad + \frac{2x^8}{15} \int_x^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^5}{1+e^{2\pi\nu}} \right) \right] \left[ \left( \frac{\nu}{x} \right)^2 - 1 \right]^{1/2}, \\
f(6; 9/2) &= \frac{5\pi}{32}x^7 - \frac{\pi}{48}x^8 \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^5}{1+e^{2\pi\nu}} \right) \Big|_{\nu=x}, \\
f(7; 9/2) &= \frac{x^8}{2(s+1/2)} + (\ln x - 11/12)x^8 \\
&\quad + \frac{x^8}{48} \int_0^\infty d\nu \left[ \frac{d}{d\nu} \left( \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{1}{\nu} \frac{d}{d\nu} \frac{\nu^6}{1+e^{2\pi\nu}} \right) \right] \ln |\nu^2 - x^2|. \tag{B.1}
\end{aligned}$$

With them, all the  $A_i(s)$  are obtained immediately and, what is very important, always in the most suitable fashion for practical evaluation (as explained before).

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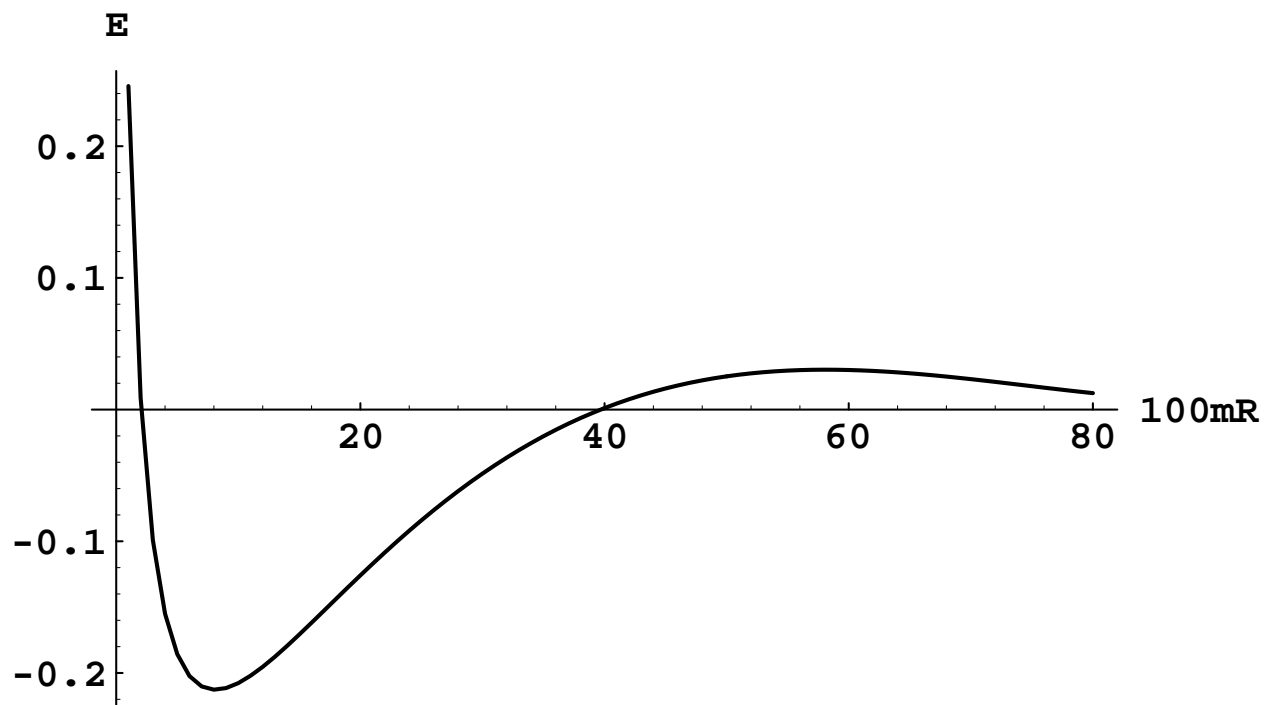


Figure 1: The energy  $E_0^{(ren)}$  as a function of the radius for a specific choice of parameters